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Special Geometry

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Abstract: $N = 2$ Supergravity*

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Contents

1. $N = 2$ Supergravity

1.1 Supersymmetric multiplets

We consider $D = 4$, $N = 2$ supergravity. The supersymmetric multiplets with spin less or equal than 2 are,

- One gravity multiplet, containing the graviton $g_{\mu\nu}$, two gravitini ψ_μ^α and one Abelian gauge field $\mathcal{A}_\mu$ known as the graviphoton.

$$(-2, -\frac{3}{2}^2, -1) + (+1, +\frac{3}{2}^2, +2) \tag{1.1}$$

- n_V vector multiplet, each consisting of one Abelian gauge field A_μ, two gaugini λ^α and one complex scalar z. The complex scalars z take values in a projective special Khler manifold $\mathcal{M}_V$ of real dimension $2n_V$.

$$(-1, -\frac{1}{2}^2, 0) + (0, +\frac{1}{2}^2, +1), \quad n_V \text{ copies} \tag{1.2}$$

- n_H hypermultiplets, each consisting of two complex scalars and two hyperinis ψ, $\tilde{\psi}$. The scalars take values in a queternionic-Khler space $\mathcal{M}_h$ of real dimension $4n_H$.

$$(-\frac{1}{2}, 0^2, \frac{1}{2}) + (-\frac{1}{2}, 0^2, \frac{1}{2}), \quad n_H \text{ copies} \tag{1.3}$$

We consider the ungauged $N = 2$ supergravity, i.e., the hypermultiplets is not charged by the vector multiplets. The special geometry describes the *vector multiplet.*

1.2 Special geometry

The coupling of the vecot multiplets, including the geometry of the scalar manifold $\mathcal{M}_V$, are conveniently described by means of a $Sp(2n_V+2)$ principal bundle ϵ over $\mathcal{M}_V$, and its associated bundle ϵ_V in the vector representation of $Sp(2n_V+2)$. The origin of the symplectic symmetry lies in electric-magnetic duality, which mixes the n_V vectors A_μ and the graviphoton $\mathcal{A}_\mu$ together with their magnetic duals. Denoting a section Ω by its coordinates (X^I, F_I), $(I=0,...,n_V+1)$, the antisymmetric product

$$\langle\Omega,\Omega'\rangle = \begin{pmatrix} X^I & F_J \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X'^J \\ F'_I \end{pmatrix} = X^I F'_I - X'^I F_I \tag{1.4}$$

The symplectic form is $\langle d\Omega, d\Omega'\rangle = dX^I \wedge dF_I$.

The geometry of $\mathcal{M}_V$ is completely determined by a choice of a holomorphic section $\Omega(z) = (X^I(z), F_I(z))$ taking value in a Lagrangian cone, i.e. a dilation invariant subspace such that $dX^I \wedge dF_I = 0$. The special geometry constraint is,

$$\partial_J X^I F_I - X^I \partial_J F_I = 0, \tag{1.5}$$

where $J = 1,...n_V+1$ and the derivatives are in the projective coordinates of $\mathcal{M}_V$. Furthermore, we may choose the

X^I as the projective coordinates, hence (1.5) is simplified to,

$$F_I = X^J \frac{\partial F_J}{\partial X_I}. \tag{1.6}$$

So we can define the prepotential $F = \frac{1}{2} X^J F_J$ such that,

$$F_I = \frac{\partial F}{\partial X^I}. \tag{1.7}$$

The prepotential is an homogeneous function of degree 2 in the X^I. The Hessian of the prepotential, $\tau_{IJ} = \partial_I \partial_J F$, is independent of X^I. Hence

$$F_I = \tau_{IJ} X^J. \tag{1.8}$$

At a generic point on $\mathcal{M}_V$, we can choose the *special coordiates* $z_i = X^i/X^0$ $(i = 1,...,n_V)$ as the holomorphic coordinates for $\mathcal{M}_V$. Once the holomorphic section $\Omega(z)$ is given, the metric on $\mathcal{M}_V$ is obtained from the Kähler potential,

$$\mathcal{K}(z^i, \bar{z}^i) = -\log\Big(i\langle\bar{\Omega},\Omega\rangle\Big) = -\log\Big(i(\bar{X}^I F_I - X^I \bar{F}_I)\Big). \tag{1.9}$$

The metric

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K} = i e^{\mathcal{K}} \langle \partial_i \Omega, \bar{\partial}_j \bar{\Omega}\rangle - \partial_i \mathcal{K} \bar{\partial}_j \mathcal{K} \tag{1.10}$$

Under a Kähler transformation, $\Omega \to e^{f(z)}\Omega$,

$$\mathcal{K} \to \mathcal{K} - f(z) - \bar{f}(\bar{z}), \tag{1.11}$$

and the metric is invariant. We may define the rescaled holomorphic section as,

$$\tilde{\Omega} = e^{\mathcal{K}/2}\Omega, \tag{1.12}$$

which transforms by a phase, $\tilde{\Omega} \to e^{(f-\bar{f})/2}\tilde{\Omega}$ under the Kähler transformation.

The derived section is defined by $U_i = D_i\tilde{\Omega} = (f_i^I, h_{iI})$, where

$$f_i^I = e^{\mathcal{K}/2} D_i X^I = e^{\mathcal{K}/2}(\partial_i X^I + \partial_i \mathcal{K} X^I) \tag{1.13}$$

$$h_{iI} = e^{\mathcal{K}/2} D_i h_I = e^{\mathcal{K}/2}(\partial_i F_I + \partial_i \mathcal{K} F_I). \tag{1.14}$$

Hence the metric is

$$g_{i\bar{j}} = -i\langle U_i, \bar{U}_{\bar{j}}\rangle. \tag{1.15}$$

1.3 4D action

The kinetic term of the $n_V + 1$ Abelian gauge fields (including the graviphoton) is $(I = 0, ..., n_V)$,

$$\mathcal{L}_{Maxwell} = -(Im\mathcal{N}_{IJ})\mathcal{F}^I \wedge \star\mathcal{F}^J + (Re\mathcal{N}_{IJ})\mathcal{F}^I \wedge \mathcal{F}^J \tag{1.16}$$

where $\mathcal{N}_{IJ}$ is defined to be

$$\mathcal{N}_{IJ} = \bar{\tau}_{IJ} + 2i\frac{(Im\tau \cdot X)_I (Im\tau \cdot X)_J}{X \cdot Im\tau \cdot X}. \tag{1.17}$$

which satisfies these relations,

$$F_I = \mathcal{N}_{IJ} X^J, \quad h_{iI} = \bar{\mathcal{N}}_{IJ} f_i^J. \tag{1.18}$$

Note that $\mathcal{N}_{IJ}$ has X dependence, so the coupling constants of the 4D action depend on the vector multiplets moduli. $Im\mathcal{N}_{IJ}$ is a negative definite matrix, as required for the positive definiteness fo the gauge kinetic terms.

We may absorb the Yang-Mills angle terms as,

$$\mathcal{L}_{Maxwell} = Im[\bar{\mathcal{N}}_{IJ}\mathcal{F}^{I-} \wedge \star\mathcal{F}^{J-}] \tag{1.19}$$

where $\mathcal{F}^{I-} = (\mathcal{F}^I - i \star \mathcal{F}^I)/\sqrt{2}$.

The dual field of $F^{I;\mu\nu}$ is, [1]

$$\mathcal{G}_I = \frac{1}{2}\frac{\partial \mathcal{L}_{Maxwell}}{\partial \mathcal{F}^I} = (Re\mathcal{N})_{IJ}\mathcal{F}^J + (Im\mathcal{N})_{IJ} \star \mathcal{F}^J \tag{1.20}$$

[1] The functional derivative should be viewed as the formal derivative in F^I, not $F^{I,\mu\nu}$.

Under the symplectic transformation,

$$\begin{pmatrix} X \\ F \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix} \tag{1.21}$$

$\mathcal{N}$ transforms as "period matrix" $\mathcal{N} \to (C + D\mathcal{N})(A + B\mathcal{N})^{-1}$, while the field strengths $(\mathcal{F}^{I-}, G_I^- = \bar{\mathcal{N}}_{IJ}\mathcal{F}_{\mu\nu}^{J-})$ transform as a symplectic vector,

$$\begin{pmatrix} \mathcal{F}^- \\ \mathcal{G}^- \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^- \\ \mathcal{G}^- \end{pmatrix}. \tag{1.22}$$

The 4D action can be rewritten as

$$\mathcal{L}_{Maxwell} = Im(G_I^- \wedge \star\mathcal{F}^{I-}), \tag{1.23}$$

which is invariant under the simplectic transformation.

1.4 Central charge

The field strength of the graviphoton is,

$$T_{\mu\nu}^- = -2ie^{\mathcal{K}/2}X^I(Im\mathcal{N})_{IJ}\mathcal{F}^{J-} = e^{\mathcal{K}/2}(X^I\mathcal{G}_I^- - F_I\mathcal{F}^{I-}). \tag{1.24}$$

The charges associated to $T_{\mu\nu}^-$ measured at infinity,

$$Z = e^{\mathcal{K}/2}(q_I X^I - p^I F_I), \tag{1.25}$$

is the central charge in $\mathcal{N} = 2$ supersymmetry algebra,

$$\{Q_\alpha^i, \bar{Q}_{\dot{\alpha}j}\} = P_\mu\sigma^\mu_{\alpha\dot{\alpha}}\delta^i_j, \quad \{Q_\alpha^i, Q_\beta^j\} = Z\epsilon^{ij}\epsilon_{\alpha\beta}, \tag{1.26}$$

where $i, j = 1, 2$. The BPS bound is,

$$M \geq |Z|. \tag{1.27}$$

2. Calabi-Yau